ARITHMETIC IN MAYA NUMERALS

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ABSTRACT

Arithmetical procedures, including addition, subtraction, multiplication, division, and square root extraction, are demonstrated using the Maya numerals. All procedures can be carried out efficiently. The Maya system is relatively unique in that it combines properties of both place-value and non-place-value numerical systems. The Babylonian system, discussed briefly, also utilizes a mixture of properties from the two systems. In order to take into account the unique hybrid characteristics of these two systems, as well as the subtractive principle of the Roman numerals, we here define a third category of numerical systems designated as mixed-place-value in type. The three types of numerical systems are compared and the advantages and disadvantages of each are mentioned. The evolutionary development of numerical systems in relation to the mathematical needs of societies is discussed.

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NUMERICAL SYSTEMS have been classified in the past into two types. The first is the place-value system represented by our own Hindu-Arabic numerals, by modern Chinese, etc. The second is the non-place-value system, represented by Minoan Linear A and B, by Egyptian hieroglyphics, by Greek acrophonic and alphabetical numerals, etc. We define here a third group of numerical systems which are hybrid in structure and for which we propose the designation mixed-place-value numerical systems. This third group is represented by the Maya, the Babylonian, and the Roman numerals. The Roman numerals constitute primarily a non-place-value system, but because of the use of the subtractive principle (e.g., IV represents four while VI represents six), they can be classified in the third group.

Using the Minoan Linear B numerals as an example, it already has been shown that non-place-value numerical systems can be efficiently utilized to carry out arithmetical calculations (Anderson 1958). By taking the subtractive principle into account, the Roman numerals can be used in the same manner as the entirely non-place-value systems (Anderson 1956). This paper will compare and contrast all three types of numerical systems and will examine one mixed-place-value system, the Maya, in detail.

THE BABYLONIAN NUMERICAL SYSTEM

The Babylonian system of numerals will be described briefly. It contains only two symbols, V and <, made by pressing a wedge-shaped stick into the ground or into clay tablets either with the tip pointing down, or to the left. The system makes use of the position of the symbol to represent its value: viz., V is 1, 60, or 3600 depending on position, while < is 10, 600, or 36,000. The number 671, for example, is represented by < V < V (600 + 60 + 10 + 1). Besides this place-value characteristic, the Babylonian system also utilizes a subtractive principle similar to that of the Roman numerals. However, the system also has a strong non-place-value characteristic in that all intermediate numbers (i.e., 2 to 9 and 11 to 59) are represented by repeating the basic symbol, rather than by unique separate symbols. VVV over VVV represents six; 33 is written as <<< VVV (10 + 10 + 10 + 1 + 1 + 1). This hybridization of place-value and non-place-value properties makes the Babylonian system a mixed-place-value notation. Unpublished studies have shown that the system can be used effectively for arithmetical calculations.

THE MAYA NUMERICAL SYSTEM

The Maya numerical system is unique. It contains only three symbols:

 zero · one — five

The Maya vigesimal system of notation is described in detail here. All numbers from 1 to 19 are written in a pure non-place-value notation; to illustrate; 8 is , 14 is , 16 is , etc. For larger numbers, however, a pure place-value component is added. Each larger Maya number is composed of sections: a lower, or first level, with one or more higher levels written above it. All symbols in each level are multiplied by their place-value factor. The first level factor is 1; the second level factor is 1 x 20 equals 20; the third level factor is 1 x 20 x 20 equals 400; and the fourth level factor is 1 x 20 x 20 x 20 equals 8000. Thus, this is a place-value system with a base of 20, i.e., a vigesimal system.

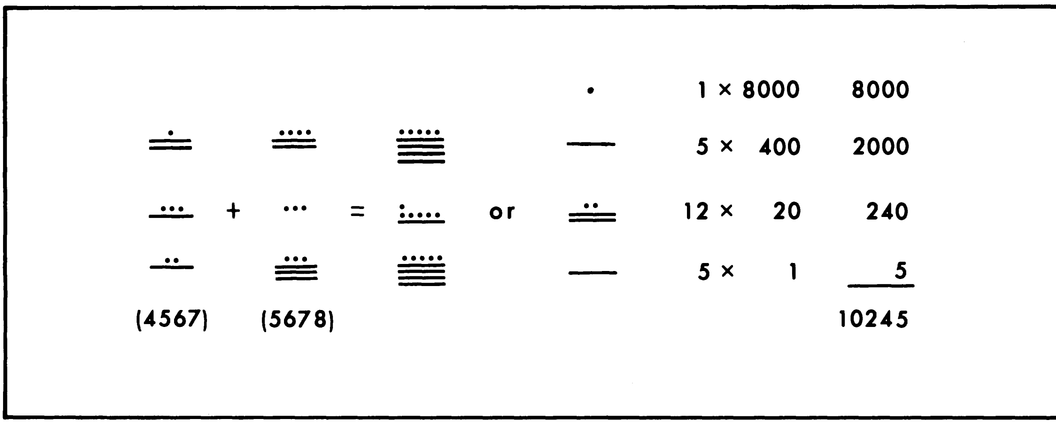

Fig. 1. Addition of 4567 plus 5678 in Maya numerals using the vigesimal notation; compare with Fig. 2 in which the same problem is solved in calendrical notation.

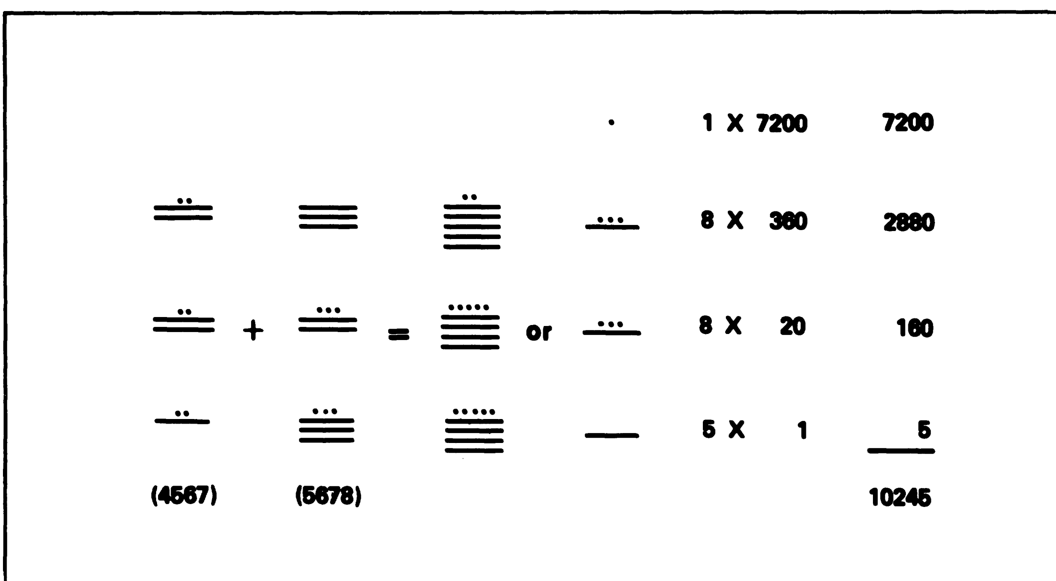

Fig. 2. Addition of 4567 plus 5678 in Maya numerals using the calendrical rather than the vigesimal notation; compare with Fig. 1.

The Maya also used, for chronology, a calendric notation which is only slightly different, and which can be utilized with equal effectiveness (compare Figs. 1 and 2). The calendrical system uses only 18 units in the second level. Consequently, each unit in the third level represents 360 (1 x 20 x 18), rather than 400; each unit in the fourth represents 7200 (1 x 18 x 20 x 20), rather than 8000. Except for the example shown in Fig. 2, we will not again refer to the calendrical notation.

To illustrate the use of the vigesimal system, represents 146. The first level is 6; the second level , is 7 x 20 equals 140. Thus, 6 plus 140, or 146. represents 60; i.e., zero plus 3 x 20. 65,432 is represented by:

8 x	8000	64000
3 x	400	1200
11 x	20	220
12 x	1	12
		65432

ADDITION AND SUBTRACTION WITH MAYA NUMERALS

Arithmetical calculations in a mixed-place-value system are only somewhat more complex than those in a pure non-place-value system. Addition and subtraction in Maya numerals still remain far less complicated than in place-value systems, such as our own Hindu-Arabic. Addition requires only the counting of symbols, care being exercised to keep symbols on their proper levels. For example, as shown in Fig. 1, to add 4,567 and 5,678, the two numbers are placed side-by-side and the total number of each symbol for each level are combined, working from the bottom level to the top, for the answer. This answer is simplified by combining symbols, since five dots is the equivalent of a dash on the same level and four dashes convert to a dot in the next higher level.

$$\begin{array}{r} 4567 \\ +\ 5678 \\ \hline 10245 \end{array}$$

This same problem is solved in Fig. 2 using the calendrical notation in order to illustrate that either notation can be used equally effectively for arithmetical calculations.

Subtraction requires only the cancelling of symbols. Since in the Maya numerical system no subtractive principle is employed (in contrast to the Roman numerals and the Babylonian notation), the procedure is simply mechanical. If there are insufficient dots in a level of the minuend, then one of the bars of that level is converted to five dots. If there are insufficient bars, then a dot from the next higher level is converted to four bars in the lower level. For example, the subtraction of 52,963 from 97,549 is shown in Fig. 3.

$$\begin{array}{r} 97549 \\ -\ 52963 \\ \hline 44586 \end{array}$$

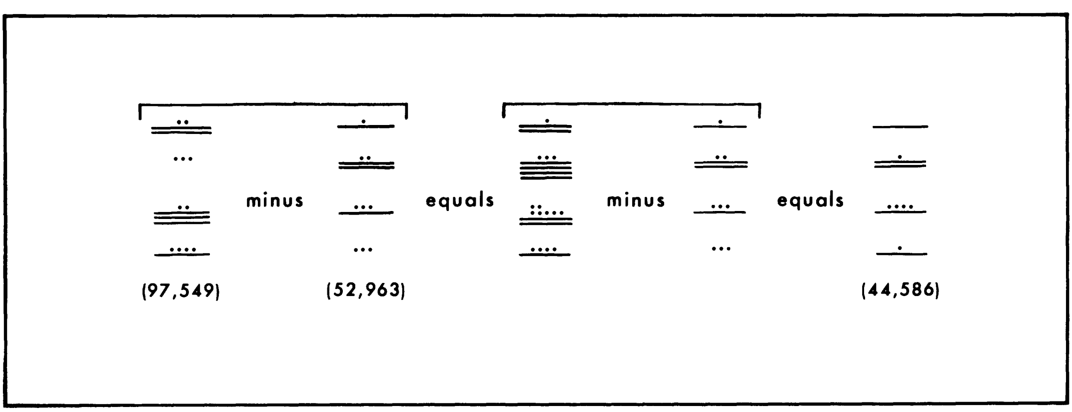

Fig. 3. Subtraction of 52,963 from 97,549 in Maya numerals. Compare with Fig. 4 in which the same problem is solved in a non-place-value system, the Minoan Linear B.

SUBTRACTION WITH MINOAN LINEAR B NUMERALS

In order to illustrate the different characteristics of the three classes of numerical systems, the above subtraction problem is represented in a non-place-value system, Minoan Linear B, in Fig. 4.

The numerals are as follows: ı one, - ten, o one hundred, ⌖ one thousand, ⌖ ten thousand.

In the Minoan system, there is no effect of position on the value of a symbol. Although, merely for convenience, the symbols are grouped with those of higher value on the left and those of lower value on the right, this arrangement is not essential. The numbers are only collections of individual symbols, and these symbols could be placed anywhere along the line. In contrast to this pure non-place-value system of notation, the addition of the place-value component to the Maya numerical system greatly reduces the number of symbols required to express large numbers. Minoan numerals required five different symbols used for a total of 86 occurrences to solve the above problem. The Maya numerals, being mixed-value with a vigesimal base, required 41 occurrences of only two (potentially three) symbols. Hindu-Arabic, our place-value system, required only 15 occurrences but of eight (potentially ten) different symbols.

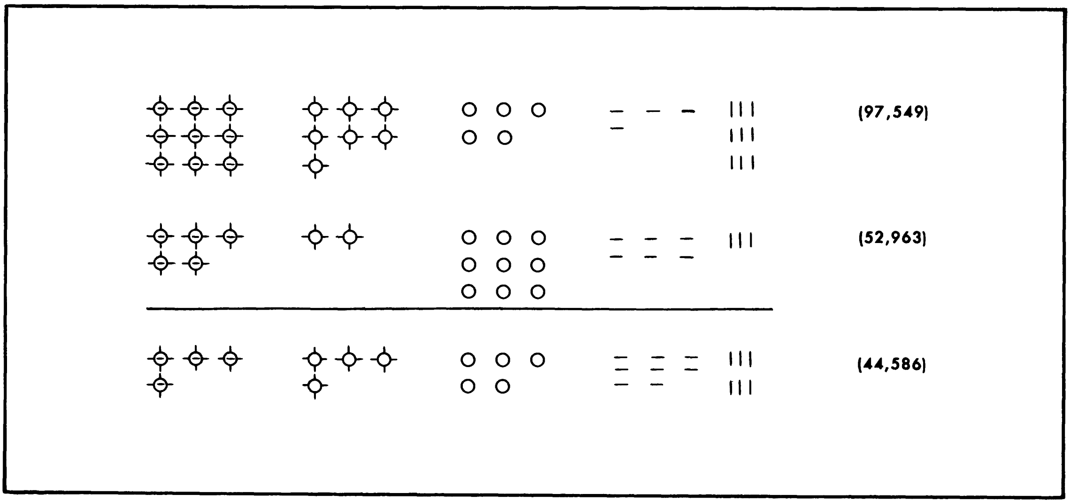

Fig. 4. Subtraction of 52,963 from 97,549 in Minoan Linear B, a non-place-value numerical system. Compare with Fig. 3.

MULTIPLICATION WITH MAYA NUMERALS

Multiplication, although in principle as easy in Maya as in Hindu-Arabic, can become difficult in practice because of the number of symbols involved in any large problem. The process itself, however, is simple. It consists of multiplying each character of the multiplicand (the top number in a Hindu-Arabic problem; the right hand set of symbols in Maya) by each character of the multiplier (the bottom number or the left hand set of symbols), being careful to keep characters in their proper levels.

The Maya multiplication tables can be determined and are shown in Fig. 5.

To illustrate the process of multiplication, $\dot{-}$ is multiplied by $\dot{-}$ in Fig. 6. The problem is set up on the left. The partial product resulting from the multiplication by each symbol of the multiplier is shown in the middle, all levels being maintained across the sheet. The partial products are added together to give the final answer as is shown on the right.

$$\begin{array}{r} 26 \\ \times\ \underline{\ \ 6} \\ 156 \end{array}$$

A more complicated problem is the multiplication of  by  and is shown in Fig. 7.

```
      572
 x    127
     4004
    1144
    572
    72644
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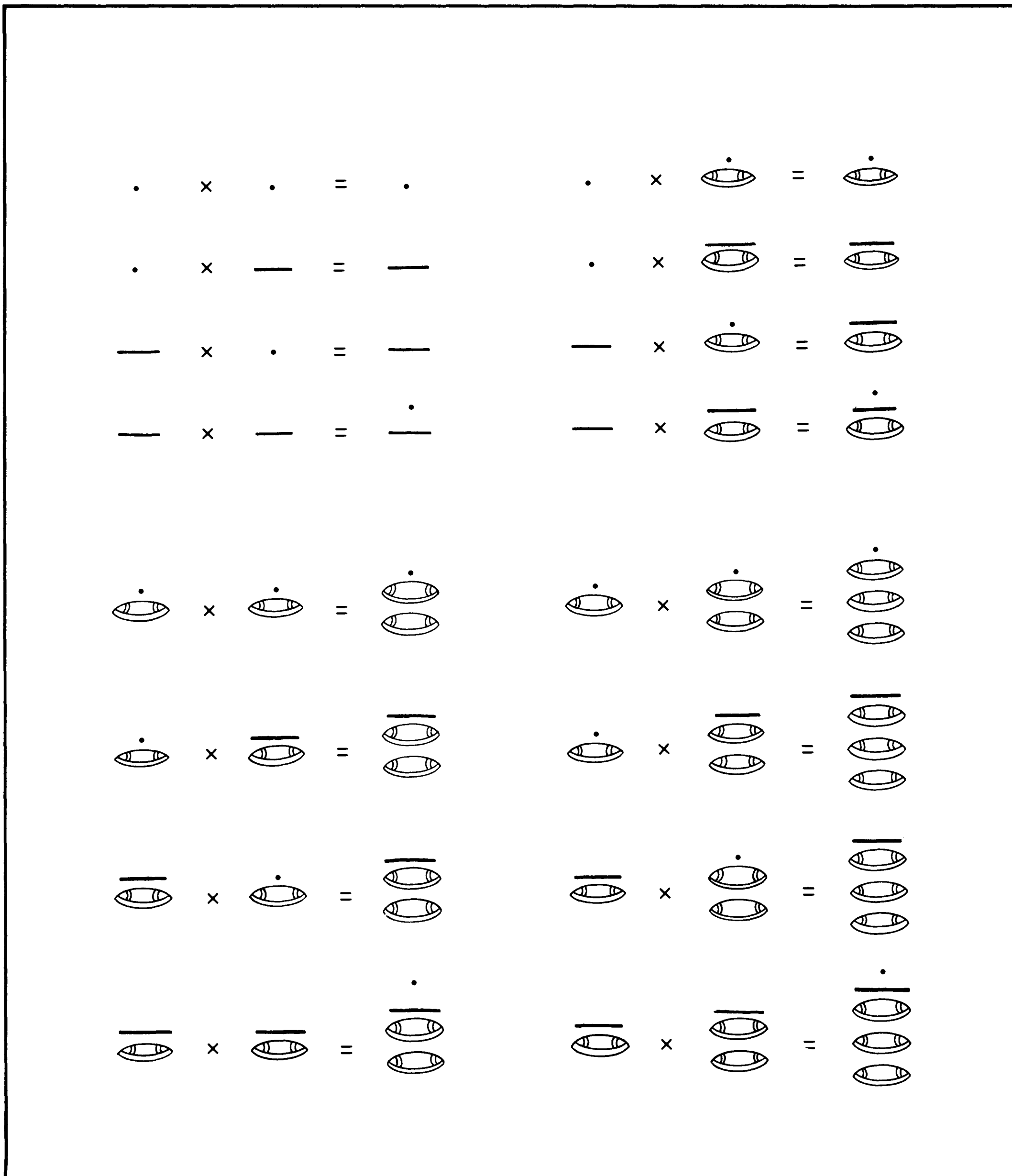

Fig. 5. The Maya multiplication tables.

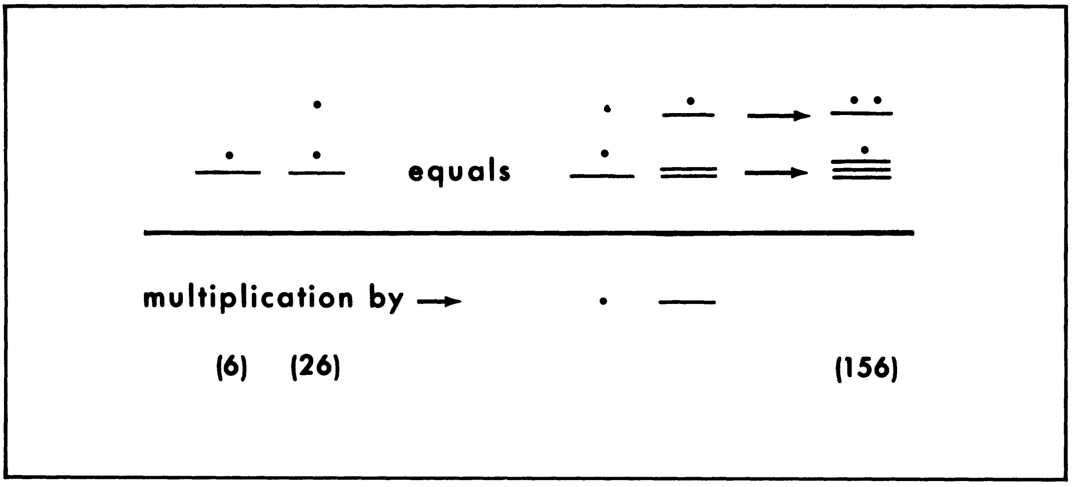

Fig. 6. Multiplication of 26 by 6 in Maya numerals.

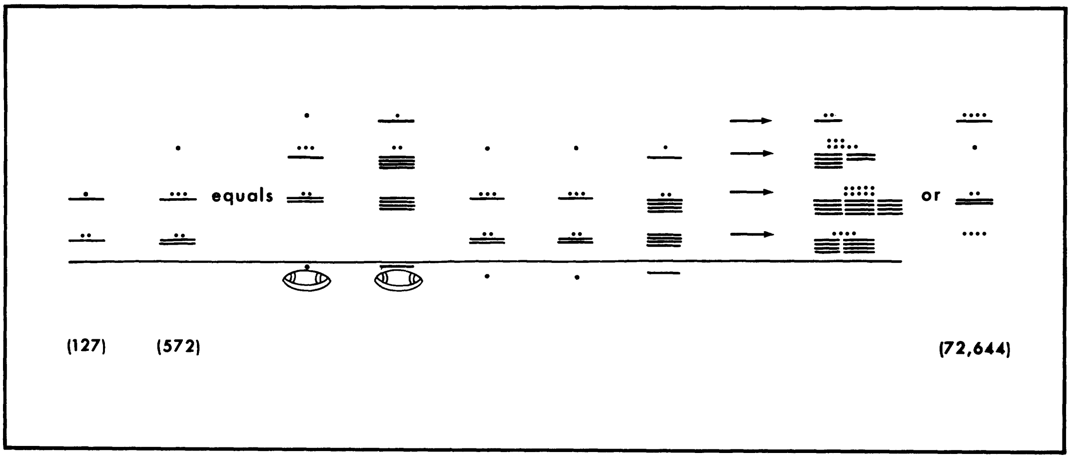

Fig. 7. Multiplication of 572 by 127 in Maya numerals.

DIVISION WITH MAYA NUMERALS

Division in Maya maintains the advantage of non-place-value systems in that it is not necessary at each step to determine the exact number of times the dividend can be divided by the divisor. For example, when dividing 63 by 3 in Hindu-Arabic, it is necessary to use 2 (actually 20) as the first number of the quotient. This is true because in a place-value system each position of the answer (ones, tens, hundreds, etc.) can be filled by only a single symbol; therefore, that symbol must be the correct one. In a non-place-value or mixed-place-value system, in contrast, more than one symbol can be present. This fact greatly simplifies the mental processes required for dividing. For example, in order to divide 246 by 6 in Maya, the problem could be set up as illustrated in Fig. 8. The levels are maintained across the page in the same manner that the decimal positions are maintained vertically when the problem is solved in Hindu-Arabic.

```
    41
 6)246
   24
     6
     6
```

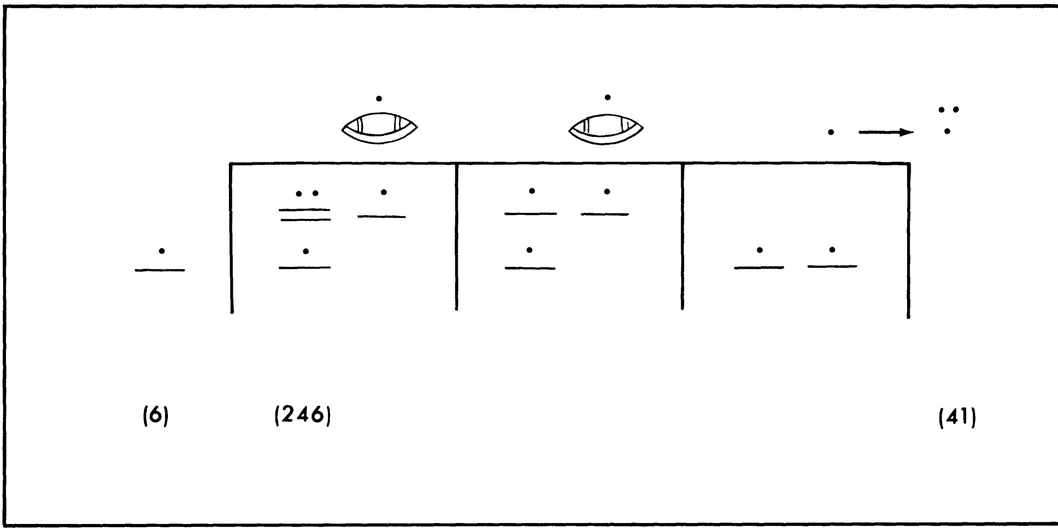

Fig. 8. Division of 246 by 6 in Maya numerals.

A more complicated problem, the division of 3591 by 133 in Maya, is shown in Fig. 9.

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       27
133)3591
    266
     931
     931
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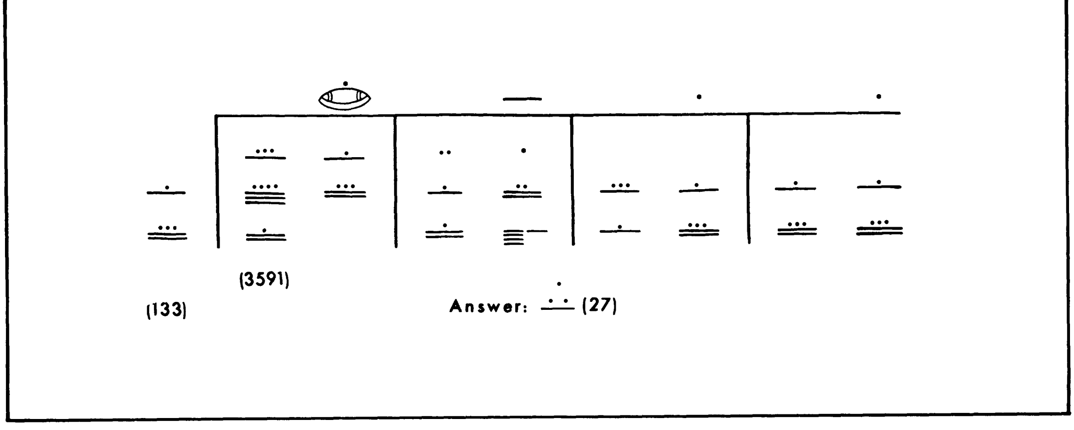

Fig. 9. Division of 3591 by 133 in Maya numerals. Compare with Fig. 10.

To illustrate the flexibility of the process of division, the above problem can be solved by making a less efficient initial choice for the quotient, as shown in Fig. 10. This example also demonstrates that two or more symbols can be added to the quotient at each step.

SQUARE ROOT EXTRACTION WITH MAYA NUMERALS

Since it has been demonstrated that Maya numerals can be used for performing the four basic processes of arithmetic (viz., addition, subtraction, multiplication, and division), this numerical system can be utilized for any arithmetical computation desired. To illustrate, the operation of extracting the square root of 225 in Maya is performed in Fig. 11. The procedure is very similar to the operation for place-value systems. At each step, a number is found which, when multiplied by the quotient doubled plus the number, can be cancelled from the remaining dividend. In a place-value system, the exact number must be used at each step. This requirement, just as in division, does not apply to Maya, although the procedure takes longer if the largest possible number is not used at each operation. The temporary multiplicand is placed at the bottom of the figure, rather than to the left (as shown below for Hindu-Arabic).

```
     √225 | 15
 1    1
     125
25   125
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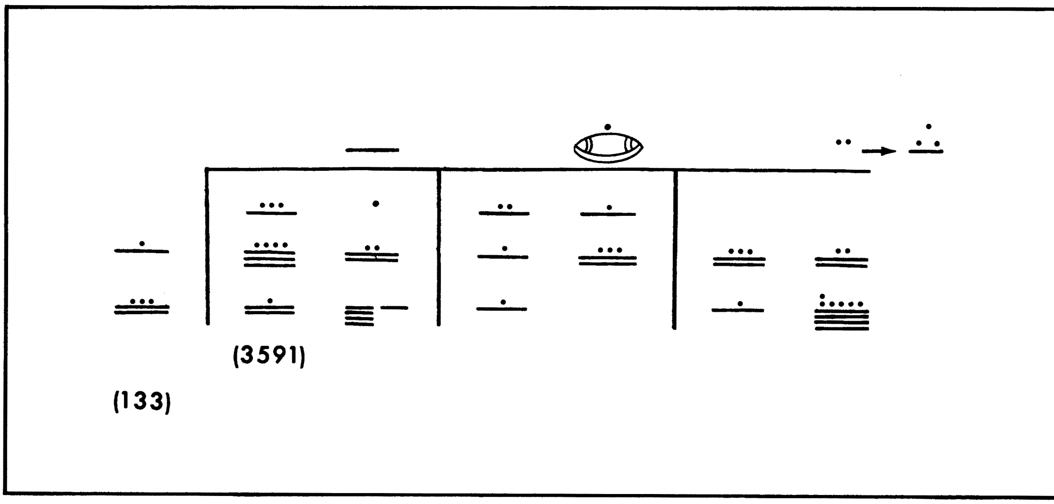

Fig. 10. Division of 3591 by 133 in Maya numerals. This is the same problem as shown in Fig. 9, but in this case, a different symbol, is used in the first step of division. The final answer is, of course, the same.

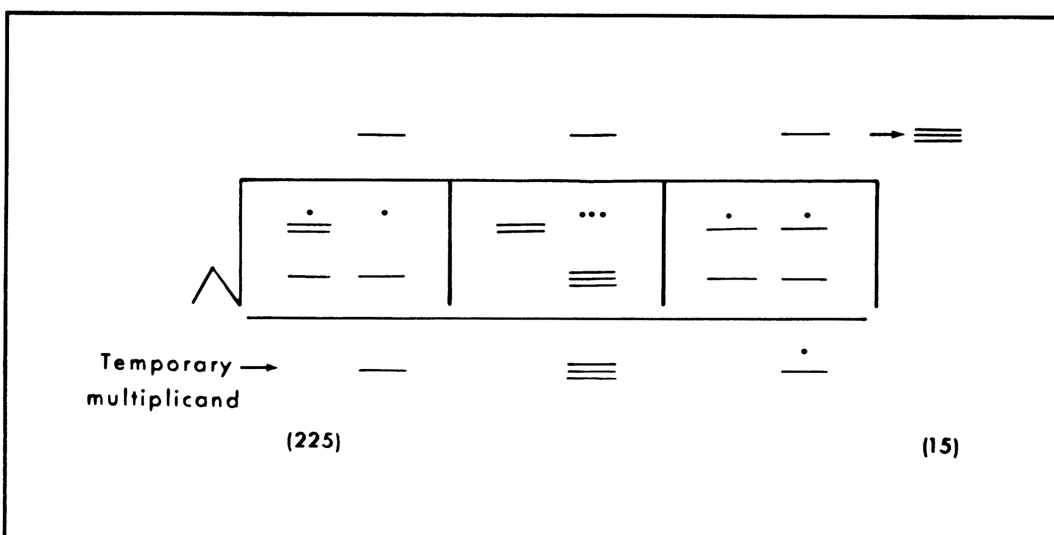

Fig. 11. Extraction of the square root of 225 in Maya numerals.

A more complicated problem, the extraction of the square root of 51,076 in Maya, is shown in Fig. 12.

	$\sqrt{51076}$	226
2	$\underline{4}$	
	110	
42	$\underline{84}$	
	2676	
446	$\underline{2676}$	

The presence of a remainder, either in division or in the extraction of a square root, creates no difficulty.

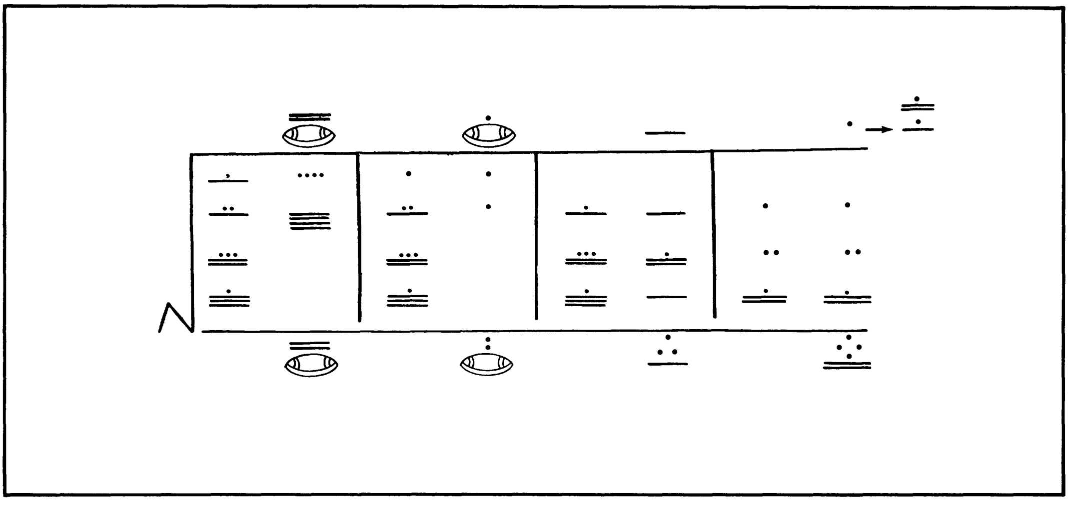

Fig. 12. Extraction of the square root of 51,076 in Maya numerals.

DISCUSSION

Like all other numerical systems so far studied, the Maya system does not offer any serious obstacles to performing arithmetical computations. The apparent difficulty lies only in the unfamiliarity of the symbols and the few, though significant, differences involved in operating a mixed-place-value or non-place-value system instead of our own familiar place-value system.

A place-value system of notation is far less cumbersome in manipulating large numbers than a non-place-value system since "place-value" means that individual symbols do not have to be repeated a number of times. Such a result is obtained, however, only by adding the concept of a symbol which varies in value depending on its position. A relatively primitive society, such as the Minoan, would probably find such a numerical system too complex for its needs. The requirements of most primitive societies are presumably satisfied by a system permitting tallying, together, perhaps, with the recording of numbers. As has been demonstrated (Anderson, 1956; 1958), addition and subtraction, the key elements of tallying, are far simpler in a non-place-value or mixed-place-value system of notation. The disadvantage of a mixed-place-value system as illustrated by the Maya, however, is the potential ease with which a symbol might be inadvertently transposed between levels. In addition, in the Babylonian and the Roman numeral systems the subtractive principle requires vigilance. Thus, the mixed-value systems, although useful in reducing the total number of symbols required for carrying out large problems, have their own inherent cumbersome properties. Pure place-value systems require considerably more abstract thinking in their use, but, in return, offer the greatest advantages, not the least of which is the ability to carry out an answer to any desired degree of accuracy.

Numerical systems appear to have undergone a process of evolution similar to societies overall. Primitive societies, for reasons outlined above, utilized simple non-place-value numerical systems. As societies became more complex, they required systems of notation which could deal with larger numbers and which could be utilized more effectively for performing complex operations. Consequently, place-value components were added to give a mixed-place value system; or else a pure place-value system was developed.

It is sometimes stated that the adoption of the place-value Hindu-Arabic system with its decimal notation and symbol for zero should be credited with enabling the scientific revolution to occur. It is just as possible, however, that the converse is true, namely, that the more sophisticated numerical systems, whether Hindu-Arabic or Maya, were developed in order to meet the more complex requirements of the society. The Hindu-Arabic system went far to satisfy the needs of the western civilization, although a duodecimal notation might have been equally or more efficient. The even more sophisticated requirements of computer mathematics utilize still another type of place-value notation: the binary system which, like the Babylonian system, has only two symbols. Unlike the Babylonian system, however, it is pure place-value with a base of two.

It is not assumed that the Mayas ever used their numerical system in the manner described in this paper. How their calculations were actually performed is not known at present. J. E. S. Thompson has examined the possible methods of computation that might have been employed for certain calendrical calculations and concludes that a counting device, perhaps resembling a simple abacus, might have been utilized (Thompson 1941, 1950). That extensive calculations were performed, particularly involving their calendrical and astronomical studies, can hardly be doubted (Thompson 1960; Satterthwaite 1947). Considering the sophistication revealed by some of these studies, it is not unreasonable to suggest that some attempt to use the numerals directly in computations might have occurred.

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